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Exercises: Charlotte Amann-Bartnick

Albert-Ludwigs-Universität Freiburg
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Model Theory

Sheet 8

Deadline: 11.12.25, 2:30 pm.

Exercise 1 (7 points).

In the language $\mathcal{L} = \{E\}$ consisting of a binary relation symbol E, let $\mathcal{K}$ be the class of finitely generated $\mathcal{L}$-structures $\mathcal{A}$ such that $E^{\mathcal{A}}$ is an equivalence relation whose classes have at most 2 elements.

a) Show that $\mathcal{K}$ is a Fraïssé class and that the Fraïssé limit M is an infinite $\mathcal{L}$-structure.

b) Provide an axiomatization T of the Fraïssé limit $\mathcal{M}$.

c) Is T totally transcendental?

d) How many types in $S_1^{\mathcal{M}}(M)$ are not isolated?

Exercise 2 (3 points).

Show that every countable small theory has a prime model.

Hint: When exactly does a theory have a prime model?

Exercise 3 (10 points).

Let $\mathcal{M}$ be a structure in a fixed language $\mathcal{L}$. We say that the element b of M is *algebraic* over the subset A of M if there is an A-instance $\varphi[x, \bar{a}]$ in $\mathrm{tp}^{\mathcal{M}}(b/A)$ such that the set

$$\varphi[M, \bar{a}] = \{c \in M \mid \mathcal{M} \models \varphi[c, \bar{a}]\}$$

is finite. The *algebraic closure* $\mathrm{acl}^{\mathcal{M}}(A)$ of the set A consists of all elements b that are algebraic over A.

a) Suppose that $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical. Show that $\mathrm{acl}^{\mathcal{M}}(A)$ is always finite, whenever A is.

b) Describe (informally) the algebraic closure of an arbitrary finite subset A of M in the case that $\mathcal{M}$ is a countable model of

i) the theory DLO of dense linear orders without endpoints.

ii) the theory of $(\mathbb{Z}, 0, s)$ as in Sheet 3, Exercise 2.

iii) the theory of an equivalence relation as in Sheet 2, Exercise 1 in the language of part d).

c) If $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, show that $\mathrm{acl}^{\mathcal{M}}(A) = \mathrm{acl}^{\mathcal{N}}(A)$ for all $A \subset M$.

d) Suppose that for every instance $\psi[x, \bar{b}]$ (with $\bar{b}$ arbitrary) in one variable the set $\psi[M, \bar{b}]$ is either finite or co-finite. Using Tarski's test, show that $\mathrm{acl}^{\mathcal{M}}(A)$ is (the universe of) an elementary substructure of $\mathcal{M}$ if A is infinite.

The exercise sheets can be handed in in pairs. Submit them in the mailbox 3.19 in the basement of the Mathematical Institute.